Quantum discord under two-side projective measurements

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The original definition of quantum discord of bipartite states was defined under one-side projective measurements, it describes quantum correlation more extensively than entanglement. Dakic, Vedral, and Brukner [Phys. Rev. Lett. **105**, 190502 (2010)] introduced a geometric measure of quantum discord, and Luo, Fu [Phys. Rev. A **82**, 034302 (2010)] simplified the expression of it. In this paper we generalize the quantum discord to the case of two-side projective measurements, and also define a geometric measure on it. Further, a simplified expression and a lower bound of this geometric measure are derived and explicit expressions are obtained for some special cases.

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I. INTRODUCTION: QUANTUM DISCORD UNDER ONE-SIDE PROJECTIVE MEASUREMENTS

Quantum correlation is one of the most striking features in quantum many-body systems. Entanglement was widely regarded as nonlocal quantum correlation and it leads to powerful applications [1, 2]. However, entanglement is not the only type of correlation useful for quantum technology. A different notion of measure, quantum discord, has also been proposed to characterize quantum correlation based on quantum measurements [3, 4]. Quantum discord captures the nonlocal correlation more general than entanglement, it can exist in some states even if entanglement does vanish. Moreover, it was shown that quantum discord might be responsible for the quantum computational effiency of some quantum computation tasks [5–7].

Recently, quantum discord has attracted increasing attention. Its evaluation involves optimization procedure, and analytical expressions are known only in a few cases [8, 9]. A witness of quantum discord for $2\times n$ states was found [10], while we have known that almost all quantum states have nonvanishing quantum discord [11]. Theoretically, the relations between quantum discord and other concepts have been discussed, such as Maxwell's demon [12, 13], completely positive maps [14], and relative entropy [15]. Also, the characteristics of quantum discord in some physical models and in information processing have been studied [16–19].

The original definition of quantum discord was given under one-side projective measurements. In this paper, we will generalize it to the case of two-side projective measurements. For clarity, we first give some notations and rules which will be used throughout this paper: Let H^A, H^B be the Hilbert spaces of quantum systems A, B, $dimH^A=m$, $dimH^B=n$. I_A, I_B, I are the identity operators on H^A, H^B and $H^A\otimes H^B$. The reduced density matrices of a state ρ^{AB} on $H^A\otimes H^B$ are $\rho^A=tr_B\rho$, $\rho^B=tr_A\rho$. For any density operators ρ,σ on a Hilbert space H, the entropy of ρ is $S(\rho)=-tr\rho\log\rho$ ($\log\rho=\log_2\rho$), the relative entropy is $S(\rho||\sigma)=tr\rho\log\rho-tr\rho\log\sigma$. It is known that $S(\rho||\sigma)\geq 0$ and $S(\rho||\sigma)=0$ only if $\rho=\sigma$. The conditional entropy of ρ^{AB} on $H^A\otimes H^B$ (with respect to A) is defined as $S(\rho^{AB})-S(\rho^A)$, and the mutual information of ρ is $S(\rho^A)+S(\rho^B)-S(\rho^{AB})$ which is nonnegative and vanishing only when $\rho^{AB}=\rho^A\otimes\rho^B$. A general measurement on ρ^{AB} is denoted by a set of operators $\Phi=\{\Phi_\alpha\}_\alpha$ on $H^A\otimes H^B$ satisfying $\sum_\alpha\Phi_\alpha\Phi_\alpha^\dagger=I$, where $\dagger$ means Hermitian adjoint, and $\{\Phi_\alpha\}_\alpha$ operate ρ^{AB} as $\widetilde{\rho^{AB}}=\sum_\alpha\Phi_\alpha\rho^{AB}\Phi_\alpha^\dagger$. When $\Phi_\alpha=A_\alpha\otimes I_B$, where A_α are operators on H^A, we say $\{A_\alpha\otimes I_B\}_\alpha$ is a one-side (with respect to subsystem A) general measurement. Moreover, if $A_\alpha=\Pi_\alpha=|\alpha\rangle\langle\alpha|$ and $\{|\alpha\rangle\}_{\alpha=1}^m$ is an orthonormal basis of H^A, we call $\{\Pi_\alpha\otimes I_B\}_\alpha$ a one-side projective measurement. Similarly, we call $\{\Pi_{\alpha\beta}\}_{\alpha,\beta}$ a two-side projective measurements, where $\Pi_{\alpha\beta}=|\alpha\rangle\langle\alpha|\otimes|\beta\rangle\langle\beta|$, and $\{|\beta\rangle\}_{\beta=1}^n$ is an orthonormal basis of H^B. For simplicity, we sometimes write $\sum_\alpha A_\alpha\otimes I_B\rho^{AB}A_\alpha^\dagger\otimes I_B=\sum_\alpha A_\alpha\rho^{AB}A_\alpha^\dagger$ by omitting identity operators. In this paper, we use $\widetilde{\rho^{AB}}$ to denote the state whose initial state are ρ^{AB} and experienced a measurement, and $\widetilde{\rho^A}=tr_B\widetilde{\rho^{AB}}$, $\widetilde{\rho^B}=tr_A\widetilde{\rho^{AB}}$.

Now recall that the quantum discord of ρ^{AB} under one-side projective measurements on A can be expressed as

$$D_A(\rho^{AB})=S(\rho^A)-S(\rho^{AB})+\inf_{\Pi_\alpha}[S(\widetilde{\rho^{AB}})-S(\widetilde{\rho^A})]. \quad (1)$$

In Eq. (1), inf is taken over all projective measurements on A. $D_B(\rho^{AB})$ is defined similarly. The intuitive meaning of Eq. (1) is that $D_A(\rho^{AB})$ is the minimal loss of conditional entropy or mutual information (since $\rho^B=\widetilde{\rho^B}$) under all projective measurements on subsystem A.

$D_A(\rho^{AB})=0$ means there is no loss of conditional entropy or mutual information for at least one projective measurement on A. Such states are called classical states

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because of this classical feature. It can be proved that

$$D_A(\rho^{AB}) = 0 \Longleftrightarrow \rho^{AB} = \sum_{\alpha=1}^{m} p_\alpha |\alpha\rangle\langle\alpha| \otimes \rho_\alpha^B, \qquad (2)$$

where, $\{|\alpha\rangle\}_{\alpha=1}^m$ is an arbitrary orthonormal set of H^A, and p_α are probabilities.

Although the set of all states ρ^{AB} satisfying $D_A(\rho^{AB}) = 0$ is not a convex set, a technical definition of geometric measure of quantum discord of ρ^{AB} under projective measurements on A can be defined as $\inf d(\rho^{AB}, \sigma^{AB})$, where d is a distance defined on density operators of $H^A \otimes H^B$, and inf is taken over all σ^{AB} with $D_A(\sigma^{AB}) = 0$.

Let $L(H^A)$ be the real linear space of all Hermitian operators on $L(H^A)$, and define the inner product $\langle X|Y\rangle = tr_A(XY)$ for any $X, Y \in L(H^A)$, then $L(H^A)$ becomes a real Hilbert space with dimension m^2. The Hilbert spaces $L(H^B)$ and $L(H^A \otimes H^B)$ are defined similarly. A geometric measure of quantum discord of ρ^{AB} under one-side projective measurements on A can be defined as [20]

$$D_A^G(\rho^{AB}) = \inf_{\sigma^{AB}} ||\rho^{AB} - \sigma^{AB}||^2, \qquad (3)$$

where $||\rho^{AB} - \sigma^{AB}||^2 = tr[(\rho^{AB} - \sigma^{AB})^2]$, inf takes all σ^{AB} that $D_A(\sigma^{AB}) = 0$. Analytical solutions of $D_A^G(\rho^{AB})$ for all 2-qubit states were obtained [20]. Moreover, it has been showed that Eq. (3) can be simplified as [21]

$$D_A^G(\rho^{AB}) = \inf_{\Pi_\alpha} ||\rho^{AB} - \Pi_\alpha \rho^{AB} \Pi_\alpha||^2. \qquad (4)$$

In this paper, we will generalize Eqs. (1)-(4) to the case of two-side projective measurements (Sec. II), and evaluate some special states (Sec. III).

II. QUANTUM DISCORD UNDER TWO-SIDE PROJECTIVE MEASUREMENTS

We can generalize the definition of quantum discord under one-side projective measurements in Eq. (1) to the case of two-side projective measurements, as [22]

$$\begin{aligned} D_{AB}(\rho^{AB}) &= S(\rho^A) + S(\rho^B) - S(\rho^{AB}) \\ &+ \inf_{\{\Pi_{\alpha\beta}\}} [S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^A}) - S(\widetilde{\rho^B})]. \qquad (5) \end{aligned}$$

Theorem 1 below states that $D_{AB}(\rho^{AB})$ is nonnegative and for what states $D_{AB}(\rho^{AB})$ vanishes.

Theorem 1. It holds that

$$D_{AB}(\rho^{AB}) \geq 0, \qquad (6)$$

$$D_{AB}(\rho^{AB}) = 0 \Longleftrightarrow \rho^{AB} = \sum_{\alpha\beta} p_{\alpha\beta} \Pi_{\alpha\beta}. \qquad (7)$$

where $\{\Pi_{\alpha\beta}\}_{\alpha\beta}$ is an arbitrary two-side projective measurement, $p_{\alpha\beta}$ are double probabilities, that is $p_{\alpha\beta} \geq 0$, and $\sum_{\alpha\beta} p_{\alpha\beta} = 1$.

To prove Eq. (6), we first establish that $\widetilde{\rho^A \otimes \rho^B} = \widetilde{\rho^A} \otimes \widetilde{\rho^B}$ under two-side projective measurements. Given two-side projective measurement $\{\Pi_{\alpha\beta}\}_{\alpha\beta}$, we expand ρ^{AB} and $\widetilde{\rho^{AB}}$ in basis $\{|\alpha\rangle\}_{\alpha=1}^m = \{|\alpha'\rangle\}_{\alpha'=1}^m$ and $\{|\beta\rangle\}_{\beta=1}^n = \{|\beta'\rangle\}_{\beta'=1}^n$ as

$$\rho^{AB} = \sum_{\alpha\alpha'\beta\beta'} \rho^{AB}_{\alpha\alpha'\beta\beta'} |\alpha\rangle\langle\alpha'| \otimes |\beta\rangle\langle\beta'|,$$

$$\widetilde{\rho^{AB}} = \sum_{\alpha\beta} \Pi_{\alpha\beta} \rho^{AB} \Pi_{\alpha\beta} = \sum_{\alpha\beta} \rho^{AB}_{\alpha\alpha\beta\beta} |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|.$$

Then it can be easily verified that $\widetilde{\rho^A \otimes \rho^B} = \widetilde{\rho^A} \otimes \widetilde{\rho^B}$.

From the monotonicity of relative entropy under general measurements [23]

$$S(\Phi\rho^{AB}||\Phi\sigma^{AB}) \leq S(\rho^{AB}||\sigma^{AB}),$$

and the relation between mutual information and relative entropy

$$S(\rho^{AB}||\rho^A \otimes \rho^B) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}).$$

Now substituting $\{\Phi\}$ by $\{\Pi_{\alpha\beta}\}_{\alpha\beta}$ and combining $\widetilde{\rho^A \otimes \rho^B} = \widetilde{\rho^A} \otimes \widetilde{\rho^B}$, it follows that $D_{AB}(\rho^{AB}) \geq 0$.

To establish Eq. (7), it is also known that $S(\Phi\rho^{AB}||\Phi\sigma^{AB}) = S(\rho^{AB}||\sigma^{AB})$ if and only if there exists a general measurement Γ such that $\Gamma\Phi\rho^{AB} = \rho^{AB}$ and $\Gamma\Phi\sigma^{AB} = \sigma^{AB}$ [24]. Then, for any two two-side projective measurements $\{\Pi_{\alpha\beta}\}$ and $\{\Pi_{\gamma\delta}\}$, if

$$\begin{aligned} \rho^{AB} &= \sum_{\gamma\delta} \Pi_{\gamma\delta} (\sum_{\alpha\beta} \Pi_{\alpha\beta} \rho^{AB} \Pi_{\alpha\beta}) \Pi_{\gamma\delta} \\ &= \sum_{\gamma\delta} \Pi_{\gamma\delta} (\sum_{\alpha\beta} \rho^{AB}_{\alpha\alpha\beta\beta} \Pi_{\alpha\beta}) \Pi_{\gamma\delta} \\ &= \sum_{\alpha\beta\gamma\delta} \rho^{AB}_{\alpha\alpha\beta\beta} |\langle\gamma|\alpha\rangle\langle\delta|\beta\rangle|^2 \Pi_{\gamma\delta}, \end{aligned}$$

then ρ^{AB} has the form of $\rho^{AB} = \sum_{\alpha\beta} p_{\alpha\beta} \Pi_{\alpha\beta}$. Conversely, if $\rho^{AB} = \sum_{\alpha\beta} p_{\alpha\beta} \Pi_{\alpha\beta}$, then $\sum_{\alpha\beta} \Pi_{\alpha\beta} \rho^{AB} \Pi_{\alpha\beta} = \rho^{AB}$. Hence Eq. (7) holds. We thus complete the proof of Theorem 1.

From Eqs. (2) and (7), we have

$$D_{AB}(\rho^{AB}) = 0 \Longleftrightarrow D_A(\rho^{AB}) = D_B(\rho^{AB}) = 0. \qquad (8)$$

The intuitive meaning of Eq. (5) is that $D_{AB}(\rho^{AB})$ is the minimal loss of mutual information under all two-side projective measurements. We see that $D_{AB}(\rho^{AB})$ captures more correlations than $D_A(\rho^{AB})$, since

$$D_{AB}(\rho^{AB}) = 0 \Rightarrow D_A(\rho^{AB}) = 0. \qquad (9)$$

In the same spirit of Eq. (3), we also define a geometric measure of quantum discord under two-side projective measurements as

$$D_{AB}^G(\rho^{AB}) = \inf_{\chi^{AB}} ||\rho^{AB} - \chi^{AB}||^2, \qquad (10)$$

where inf takes all χ^{AB} that $D_{AB}(\chi^{AB})=0$. From Eqs. (2), (3), (7), (10), it can be easily found that

$$D_{AB}^G(\rho^{AB}) \geq \max\{D_A^G(\rho^{AB}), D_B^G(\rho^{AB})\}. \quad (11)$$

The Theorem 2 below will simplify Eq. (10).

Theorem 2. $D_{AB}^G(\rho^{AB})$ is defined in Eq. (10), then

$$D_{AB}^G(\rho^{AB}) = \inf_{\{\Pi_{\alpha\beta}\}} ||\rho^{AB} - \sum_{\alpha\beta}\Pi_{\alpha\beta}\rho^{AB}\Pi_{\alpha\beta}||^2$$
$$= tr[(\rho^{AB})^2] - \sup_{\{\Pi_{\alpha\beta}\}} ||\sum_{\alpha\beta}\Pi_{\alpha\beta}\rho^{AB}\Pi_{\alpha\beta}||^2, \quad (12)$$

where inf and sup take over all two-side projective measurements $\{\Pi_{\alpha\beta}\}$.

Proof: For any χ^{AB} that $D_{AB}(\chi^{AB})=0$, suppose

$$\chi^{AB} = \sum_{\alpha\beta} p_{\alpha\beta}|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|.$$

We expand ρ^{AB} in basis $\{|\alpha\rangle\} = \{|\alpha'\rangle\}$ and $\{|\beta\rangle\} = \{|\beta'\rangle\}$ as

$$\rho^{AB} = \sum_{\alpha\alpha'\beta\beta'} \rho^{AB}_{\alpha\alpha'\beta\beta'}|\alpha\rangle\langle\alpha'| \otimes |\beta\rangle\langle\beta'|.$$

Hence,

$$||\rho^{AB} - \chi^{AB}||^2$$
$$= tr[(\rho^{AB})^2] - 2\sum_{\alpha\beta} p_{\alpha\beta}\rho^{AB}_{\alpha\alpha\beta\beta} + \sum_{\alpha\beta} p^2_{\alpha\beta}$$
$$= tr[(\rho^{AB})^2] - \sum_{\alpha\beta}(\rho^{AB}_{\alpha\alpha\beta\beta})^2 + \sum_{\alpha\beta}(p_{\alpha\beta} - \rho^{AB}_{\alpha\alpha\beta\beta})^2.$$

By choosing $p_{\alpha\beta} = \rho^{AB}_{\alpha\alpha\beta\beta}$, i.e., $\chi^{AB} = \Pi_{\alpha\beta}\rho^{AB}\Pi_{\alpha\beta}$, we then attain Theorem 2.

We would rather like to give another expression of Theorem 2 and a lower bound of $D_{AB}^G(\rho^{AB})$ follws from it, that is Theorem $2'$ below.

Theorem $2'$. $D_{AB}^G(\rho^{AB})$ is defined in Eq. (10), then

$$D_{AB}^G(\rho^{AB}) = tr(CC^t) - \sup_{AB} tr(ACB^tBC^tA^t), \quad (13)$$

$$D_{AB}^G(\rho^{AB}) \geq tr(CC^t) - \sum_{k=1}^{\min\{m,n\}} \lambda_k. \quad (14)$$

Where λ_k are the eigenvalues of CC^t listed in decreasing order (counting multiplicity), t denotes transpose. The meanings of matrices A, B, C as follows: given orthonormal bases $\{X_i\}_{i=1}^{m^2}$ for $L(H^A)$ and $\{Y_j\}_{j=1}^{n^2}$ for $L(H^B)$. Let $\rho^{AB} = \sum_{ij} C_{ij}X_i \otimes Y_j$, matrix $C = (C_{ij})$. For any orthonormal bases $\{|\alpha\rangle\}_{\alpha=1}^m$ for H^A and $\{|\beta\rangle\}_{\beta=1}^n$ for H^B, let $|\alpha\rangle\langle\alpha| = \sum_{i=1}^{m^2} A_{\alpha i}X_i$, $|\beta\rangle\langle\beta| = \sum_{j=1}^{n^2} B_{\beta j}Y_j$ and matrices $A = (A_{\alpha i}), B = (B_{\beta j})$.

To prove Eq. (13), note that $A_{\alpha i} = tr(X_i|\alpha\rangle\langle\alpha|) = \langle\alpha|X_i|\alpha\rangle$, $B_{\beta j} = tr(Y_j|\beta\rangle\langle\beta|) = \langle\beta|Y_j|\beta\rangle$, thus

$$D_{AB}^G(\rho^{AB}) = tr[(\rho^{AB})^2] - \sup_{\{\Pi_{\alpha\beta}\}} ||\sum_{\alpha\beta}\Pi_{\alpha\beta}\rho^{AB}\Pi_{\alpha\beta}||^2$$
$$= \sum_{ij} C^2_{ij} - \sup_{\{\Pi_{\alpha\beta}\}} ||\sum_{ij\alpha\beta} C_{ij}\langle\alpha|X_i|\alpha\rangle\langle\beta|Y_j|\beta\rangle|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|||^2$$
$$= \sum_{ij} C^2_{ij} - \sup_{AB}\sum_{\alpha\beta}(\sum_{ij} A_{\alpha i}C_{ij}B_{\beta j})^2$$
$$= tr(CC^t) - \sup_{AB} tr(ACB^tBC^tA^t).$$

A brief proof of inequality (14) is: since [21]

$$D_A^G(\rho^{AB}) \geq tr(CC^t) - \sum_{k=1}^m \lambda_k,$$
$$D_B^G(\rho^{AB}) \geq tr(CC^t) - \sum_{k=1}^n \lambda_k,$$

together with $D_{AB}^G(\rho^{AB}) \geq \max\{D_A^G(\rho^{AB}), D_B^G(\rho^{AB})\}$, thus inequality (14) is readily true.

III. EXAMPLES

In this section let us consider some examples which allow explicit results.

Example 1. For the $m \times m$ Werner state

$$\rho^{AB} = \frac{m-x}{m^3-m}I + \frac{mx-1}{m^3-m}F, \quad x \in [-1,1],$$

with $F = \sum_{kl}|k\rangle\langle l| \otimes |l\rangle\langle k|$. Note that $F^2 = I$, $trF = m$. For any two-side projective measurement $\{\Pi_{\alpha\beta}\}$,

$$\sum_{\alpha\beta}\Pi_{\alpha\beta}\rho^{AB}\Pi_{\alpha\beta}$$
$$= \frac{m-x}{m^3-m}I + \frac{mx-1}{m^3-m}\sum_{\alpha\beta kl}\langle\alpha|k\rangle\langle l|\alpha\rangle\langle\beta|l\rangle\langle k|\beta\rangle\Pi_{\alpha\beta}$$
$$= \frac{m-x}{m^3-m}I + \frac{mx-1}{m^3-m}\sum_{\alpha\beta}|\langle\alpha|\beta\rangle|^2\Pi_{\alpha\beta}.$$

Thus, applying Lagrangian multipliers method, we get

$$D_{AB}^G(\rho^{AB}) = \frac{(mx-1)^2}{m(m-1)(m+1)^2}.$$

That is, $D_{AB}^G(\rho^{AB}) = D_A^G(\rho^{AB})$. [21]

A werner state is separable if and only if $x \in (0,1]$, but its geometric measures of quantum discords vanish if and only if $x = 1/m$.

Example 2. For the $m \times m$ isotropic state

$$\rho^{AB} = \frac{1-x}{m^2-1}I + \frac{m^2x-1}{m^2-1}M, \quad x \in [0,1],$$

with $M = \frac{1}{m}\sum_{kl}|k\rangle\langle l| \otimes |k\rangle\langle l|$. Note that $M^2 = M$, and $trM = 1$. For any two-side projective measurement $\{\Pi_{\alpha\beta}\}$,

$$\sum_{\alpha\beta}\Pi_{\alpha\beta}\rho^{AB}\Pi_{\alpha\beta}$$

$$= \frac{1-x}{m^2-1}I + \frac{m^2x-1}{m^2-1}\frac{1}{m}\sum_{\alpha\beta kl}\langle\alpha|k\rangle\langle l|\alpha\rangle\langle\beta|k\rangle\langle l|\beta\rangle\Pi_{\alpha\beta}$$
$$= \frac{1-x}{m^2-1}I + \frac{m^2x-1}{m^2-1}\frac{1}{m}\sum_{\alpha\beta}|\langle\alpha|\beta'\rangle|^2\Pi_{\alpha\beta}.$$

here $|\beta'\rangle = |\beta\rangle^*$ is the complex conjugate of $|\beta\rangle$. Using the similar techniques in example 1 will yield

$$D^G_{AB}(\rho^{AB}) = \frac{(m^2x-1)^2}{m(m-1)(m+1)^2}.$$

That is $D^G_{AB}(\rho^{AB}) = D^G_A(\rho^{AB})$. [21]

Recall that an isotropic state is separable if and only if $x \in [0, 1/m]$, but its geometric measures of quantum discords vanish if and only if $x = 1/m^2$.

Example 3. For any two-qubit state

$$\rho = \frac{1}{4}(I + \sum_{i=1}^{3} x_i\sigma_i \otimes I_2 + \sum_{j=1}^{3} y_j I_1 \otimes \sigma_j + \sum_{i,j=1}^{3} T_{ij}\sigma_i \otimes \sigma_j)$$
$$= \frac{1}{2}(X_0 \otimes Y_0 + \sum_{i=1}^{3} x_i X_i \otimes Y_0 + \sum_{j=1}^{3} y_j X_0 \otimes Y_j + \sum_{i,j=1}^{3} T_{ij} X_i \otimes Y_j).$$

Where $\{\sigma_i\}$ are the Pauli matrices, $\{X_0, X_1, X_2, X_3\} = \frac{1}{\sqrt{2}}\{I_1, \sigma_1, \sigma_2, \sigma_3\}$, $\{Y_0, Y_1, Y_2, Y_3\} = \frac{1}{\sqrt{2}}\{I_2, \sigma_1, \sigma_2, \sigma_3\}$. Note that $tr\sigma_i = 0$ and $tr(\sigma_i\sigma_j) = 2\delta_{ij}$, hence $\{X_0, X_1, X_2, X_3\}$ is an orthonormal basis for $L(H^A)$, and $\{Y_0, Y_1, Y_2, Y_3\}$ is an orthonormal basis for $L(H^B)$. For any orthonormal basis $\{|\alpha\rangle\}_{\alpha=1}^2$ of H^A, $|\alpha\rangle\langle\alpha| \in L(H^A)$, we can write $|\alpha\rangle\langle\alpha|$ as

$$|1_A\rangle\langle 1_A| = \frac{1}{\sqrt{2}}(X_0 + a_1X_1 + a_2X_2 + a_3X_3),$$
$$|2_A\rangle\langle 2_A| = \frac{1}{\sqrt{2}}(X_0 - a_1X_1 - a_2X_2 - a_3X_3).$$

Here, $\mathbf{a} = (a_1, a_2, a_3)$ is a real vector with $\|\mathbf{a}\| = \sum_{i=1}^3 a_i^2 = 1$. Similarly, for any orthonormal basis $\{|\beta\rangle\}_{\alpha=1}^2$ of H^B, $|\beta\rangle\langle\beta| \in L(H^B)$, we write $|\beta\rangle\langle\beta|$ as

$$|1_B\rangle\langle 1_B| = \frac{1}{\sqrt{2}}(Y_0 + b_1Y_1 + b_2Y_2 + b_3Y_3),$$
$$|2_B\rangle\langle 2_B| = \frac{1}{\sqrt{2}}(Y_0 - b_1Y_1 - b_2Y_2 - b_3Y_3).$$

Here, $\mathbf{b} = (b_1, b_2, b_3)$ is a real vector with $\|\mathbf{b}\| = 1$.

Thus, from Eq. (13), direct calculation shows that

$$D^G_{AB}(\rho^{AB}) = \frac{1}{4}[||\mathbf{x}||^2 + ||\mathbf{y}||^2 + tr(TT^t)] - \frac{1}{4}\sup_{\mathbf{ab}}[(\mathbf{a}\cdot\mathbf{x})^2 + (\mathbf{b}\cdot\mathbf{y})^2 + (\mathbf{a}T\mathbf{b}^t)^2]. \quad (15)$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, $T = (T_{ij})$.

In particular,

(i) if $T = 0$, then $D^G_{AB}(\rho^{AB}) = 0$;

(ii) if $\mathbf{x} = \mathbf{y} = 0$, that is $\rho^A = I_1$ and $\rho^B = I_2$, then $D^G_{AB}(\rho^{AB}) = \frac{1}{4}[tr(TT^t) - \lambda_{\max}]$, with $\lambda_{\max}$ being the largest eigenvalue of TT^t;

(iii) if $T_{ij} = x_iy_j$, that is $\rho^{AB} = \rho^A \otimes \rho^B$, then $D^G_{AB}(\rho^{AB}) = 0$.

IV. SUMMARY

The original definition of quantum discord $D_A(\rho^{AB})$ can be generalized to the case of two-side projective measurements by defining $D_{AB}(\rho^{AB})$ as the minimal loss of mutual information under all two-side projective measurements. We derived the set of all states that $D_{AB}(\rho^{AB})$ vanishes, and defined a geometric measure $D^G_{AB}(\rho^{AB})$ due to this set. A simplified variational expression and a lower bound of $D^G_{AB}(\rho^{AB})$ have been obtained, and some special cases allows explicit expressions.

It was shown that $D_{AB}(\rho^{AB})$ captures more correlations than $D_A(\rho^{AB})$. It is interesting to point out the containment relations below

$$\rho^{AB} = \rho^A \otimes \rho^B \Rightarrow D_{AB}(\rho^{AB}) = 0 \Rightarrow D_A(\rho^{AB}) = 0 \Rightarrow \rho^{AB} \text{ is separable.} \quad (16)$$

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Geometric measure of quantum discord over two-sided projective measurements

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Abstract

The original definition of quantum discord of bipartite states was defined over one-sided projective measurements, it describes quantum correlation more extensively than entanglement. Dakic, Vedral, and Brukner [Phys. Rev. Lett. 105 (2010) 190502] introduced a geometric measure for this quantum discord, and Luo, Fu [Phys. Rev. A 82 (2010) 034302] simplified the variation expression of it. In this paper we introduce a geometric measure for the quantum discord over two-sided projective measurements. A simplified expression and a lower bound of this geometric measure are derived and explicit expressions are obtained for some special cases.

Keywords: quantum discord, two-sided projective measurement, geometric measure

1. Introduction

Quantum entanglement is by far the most famous and best studied kind of quantum correlation, and leads to powerful applications [1, 2]. While interest remains strong, recent researches have explored another quantum correlation other than entanglement, called quantum discord [3, 4], which may be employed as alternative resources for quantum technology [5–7]. From theoretic points of view, operational interpretations of quantum discord have been proposed, the links between quantum discord with other concepts have been discussed, such as Maxwell's demon [8, 9], completely positive maps [10], and relative entropy [11]. At the same time, the characteristics of quantum discord in some physical models and in information processing have been studied [12–17].

But the awkward situation is, till now the analytical expressions of quantum discord are found only for few special states [18–22]. The problem arises from the variation expression of original definition of quantum discord. Analytical expression is very useful for investigating the dynamics in physical systems [12, 23, 24]. Very recently, Dakic, Vedral, and Brukner introduced [25] a geometric measure for quantum discord. As one of the most striking results of this measure, they obtained [25] the analytical expression for any two qubits states. Also, Luo and Fu [26] simplified the expression of this geometric measure, and derived a lower bound for any quantum state.

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In the same spirit, in this paper we introduce a geometric measure for the quantum discord over two-sided projective measurements (Sec.3). We also simplify the expression and provide a lower bound for this geometric measure (Sec.4). As examples, we derive some explicit expressions of this geometric measure for some special quantum states (Sec.5).

2. Geometric measure of quantum discord over one-sided projective measurements

For clarity, we first give some definitions about quantum entropy, conditional entropy, mutual information and projective measurement. Let H^A, H^B be the Hilbert spaces of quantum systems A, B, respectively, with $dimH^A = n_A$, $dimH^B = n_B$. I_A, I_B, I_{AB} are the identity operators on H^A, H^B and $H^A \otimes H^B$. The reduced density matrices of a state ρ^{AB} on $H^A \otimes H^B$ are $\rho^A = tr_B \rho^{AB}$, $\rho^B = tr_A \rho^{AB}$. For density operators ρ, σ on a Hilbert space H, the entropy of ρ is defined as $S(\rho) = -tr(\rho \log \rho)$ ($\log \rho = \log_2 \rho$), the conditional entropy of ρ^{AB} (with respect to A) is defined as $S(\rho^{AB}) - S(\rho^A)$. The mutual information of ρ^{AB} is defined as $S(\rho^A) + S(\rho^B) - S(\rho^{AB})$, which is nonnegative, and vanishing only when $\rho^{AB} = \rho^A \otimes \rho^B$ ([1], 11.3.4). A general measurement on ρ^{AB} is denoted by a set of operators $\Phi = \{\Phi_\alpha\}_\alpha$ satisfying $\sum_\alpha \Phi_\alpha^\dagger \Phi_\alpha = I_{AB}$, here $\dagger$ denotes Hermitian adjoint, and $\{\Phi_\alpha\}_\alpha$ performs ρ^{AB} as $\widetilde{\rho^{AB}} = \sum_\alpha \Phi_\alpha \rho^{AB} \Phi_\alpha^\dagger$. When $\Phi_\alpha = \Pi_\alpha \otimes I_B$, where $\Pi_\alpha = |\alpha\rangle\langle\alpha|$ and $\{|\alpha\rangle\}_{\alpha=1}^{n_A}$ is an orthonormal basis for H^A, we call $\{\Pi_\alpha \otimes I_B\}_\alpha$ a one-sided projective measurement. We call $\{\Pi_{\alpha\beta}\}_{\alpha\beta}$ a two-sided projective measurements, where $\Pi_{\alpha\beta} = |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|$, and $\{|\beta\rangle\}_{\beta=1}^{n_B}$ is an orthonormal basis of H^B. For simplicity, we sometimes simply write $\Pi_\alpha \otimes I_B$ as Π_α. We use $\widetilde{\rho^{AB}}$ to denote the state its initial state is ρ^{AB} and experienced a measurement, and $\widetilde{\rho^A} = tr_B \widetilde{\rho^{AB}}, \widetilde{\rho^B} = tr_A \widetilde{\rho^{AB}}$.

Now recall that the original definition of quantum discord of ρ^{AB} was defined over one-sided projective measurements (with respect to A) as [3]

$$D_A(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \inf_{\{\Pi_\alpha \otimes I_B\}_\alpha} [\Sigma_\alpha p_\alpha S(\widetilde{\rho_\alpha^B}/p_\alpha)], \tag{1}$$

where inf is taken over all projective measurements on A, $\widetilde{\rho_\alpha^B} = tr_A(\Pi_\alpha \rho^{AB} \Pi_\alpha)$, $p_\alpha = tr_B \widetilde{\rho_\alpha^B}$. $D_B(\rho^{AB})$ can be defined similarly.

By the joint entropy theorem ([1], 11.3.2), Eq.(1) can be rewritten as [3]

$$D_A(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \inf_{\{\Pi_\alpha \otimes I_B\}_\alpha} [S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^A})]. \tag{2}$$

A state ρ^{AB} satisfying $D_A(\rho^{AB}) = 0$ is called a classical state, it can be proved that [3]

$$D_A(\rho^{AB}) \geq 0, \tag{3}$$

$$D_A(\rho^{AB}) = 0 \Longleftrightarrow \rho^{AB} = \sum_{\alpha=1}^{n_A} p_\alpha |\alpha\rangle\langle\alpha| \otimes \rho_\alpha^B, \tag{4}$$

where, $\{|\alpha\rangle\}_{\alpha=1}^{n_A}$ is an arbitrary orthonormal set of H^A, $p_\alpha \geq 0$, $\sum_{\alpha=1}^{n_A} p_\alpha = 1$, ρ_α^B are density operators on H^B.

Although the set of all states ρ^{AB} satisfying $D_A(\rho^{AB}) = 0$ is not a convex set, a technical definition of geometric measure of quantum discord of ρ^{AB} over one-sided projective measurements on A can be defined as

$$D_A^G(\rho^{AB}) = \inf_{\sigma^{AB}} d(\rho^{AB}, \sigma^{AB}), \tag{5}$$

where d is a distance defined on density operators on $H^A \otimes H^B$, and inf runs over all σ^{AB} with $D_A^P(\sigma^{AB}) = 0$. $D_B^G(\rho^{AB})$ can be defined similarly. One of such geometric measure is as follows.

Let $L(H^A)$ be the real linear space of all Hermitian operators on H^A , and define the inner product $\langle X|X'\rangle = tr_A(XX')$ for any $X, X' \in L(H^A)$, then $L(H^A)$ becomes a real Hilbert space with dimension n_A^2. The Hilbert spaces $L(H^B)$ and $L(H^A \otimes H^B)$ are defined similarly. A geometric measure of quantum discord of ρ^{AB} over one-sided projective measurements on A can then be defined as [25]

$$D_A^G(\rho^{AB}) = \inf_{\sigma^{AB}} ||\rho^{AB} - \sigma^{AB}||^2, \tag{6}$$

where $||\rho^{AB} - \sigma^{AB}||^2 = tr[(\rho^{AB} - \sigma^{AB})^2]$, inf takes all σ^{AB} that $D_A^P(\sigma^{AB}) = 0$. Some analytical solutions of $D_A^G(\rho^{AB})$ were obtained [25]. Moreover, it has been shown that Eq.(6) can be simplified as [26]

$$D_A^G(\rho^{AB}) = \inf_{\{\Pi_\alpha\}_\alpha} ||\rho^{AB} - \sum_\alpha \Pi_\alpha \rho^{AB} \Pi_\alpha||^2. \tag{7}$$

3. Geometric measure of quantum discord over two-sided projective measurements

In this section, we propose a geometric measure under two-sided projective measurements.

The original definition of quantum discord over one-sided projective measurements in Eq.(1) or Eq.(2) has the intuitive physical meaning that $D_A(\rho^{AB})$ is the minimal loss of mutual information or conditional entropy due to all one-sided projective measurements. A direct way to define the quantum discord over two-sided projective measurements then is [3, 27]

$$D_{AB}(\rho^{AB}) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}) + \inf_{\{\Pi_{\alpha\beta}\}_{\alpha\beta}} [S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^A}) - S(\widetilde{\rho^B})]. \tag{8}$$

Where, inf takes all two-sided projective measurements. By the experiences of optimization about Eq.(1) or Eq.(2), it seems that Eq.(8) will be very difficult to optimize excepting some very special states. So, we introduce a geometric measure of it, just as what have done in the one-sided case [25, 26]. To do so, we first prove that $D_{AB}(\rho^{AB})$ in Eq.(8) is nonnegative for any state (then $D_{AB}(\rho^{AB})$ is a valid measure), and next we need to find the set of all states ρ^{AB} that $D_{AB}(\rho^{AB}) = 0$. This is Theorem 1 below.

Theorem 1. It holds that

$$\begin{aligned} D_{AB}(\rho^{AB}) &\geq 0, & (9)\\ D_{AB}(\rho^{AB}) &= 0 \Longleftrightarrow \rho^{AB} = \sum_{\alpha\beta} p_{\alpha\beta} \Pi_{\alpha\beta}, & (10) \end{aligned}$$

where $\{\Pi_{\alpha\beta}\}_{\alpha\beta}$ is an arbitrary two-sided projective measurement, $\{p_{\alpha\beta}\}_{\alpha\beta}$ is an arbitrary probability distrbution, that is $p_{\alpha\beta} \geq 0$, $\sum_{\alpha\beta} p_{\alpha\beta} = 1$.

proof. Given a two-sided projective measurement $\{\Pi_{\alpha\beta}\}_{\alpha\beta}$, notice that

$$\widetilde{\rho^{AB}} = \sum_{\alpha\beta} \Pi_{\alpha\beta} \rho^{AB} \Pi_{\alpha\beta} = \sum_\beta I_A \otimes \Pi_\beta (\overline{\rho_1^{AB}}) I_A \otimes \Pi_\beta, \tag{11}$$

where

$$\overline{\rho_1^{AB}} = \sum_\alpha \Pi_\alpha \otimes I_B \rho^{AB} \Pi_\alpha \otimes I_B. \tag{12}$$

We expand ρ^{AB} , $\overline{\rho_1^{AB}}$, $\widetilde{\rho^{AB}}$ and their reduced density operators in the bases $\{|\alpha\rangle\}_{\alpha=1}^{n_A} = \{|\alpha'\rangle\}_{\alpha'=1}^{n_A}$ and $\{|\beta\rangle\}_{\beta=1}^{n_B} = \{|\beta'\rangle\}_{\beta'=1}^{n_B}$ as

$$\rho^{AB} = \sum_{\alpha\alpha'\beta\beta'} \rho^{AB}_{\alpha\alpha'\beta\beta'} |\alpha\rangle\langle\alpha'| \otimes |\beta\rangle\langle\beta'|, \tag{13.1}$$

$$\rho^{A} = \sum_{\alpha\alpha'\beta} \rho^{AB}_{\alpha\alpha'\beta\beta} |\alpha\rangle\langle\alpha'|, \tag{13.2}$$

$$\rho^{B} = \sum_{\alpha\beta\beta'} \rho^{AB}_{\alpha\alpha\beta\beta'} |\beta\rangle\langle\beta'|, \tag{13.3}$$

$$\overline{\rho_1^{AB}} = \sum_{\alpha} \Pi_\alpha \rho^{AB} \Pi_\alpha = \sum_{\alpha\beta\beta'} \rho^{AB}_{\alpha\alpha\beta\beta'} |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta'|, \tag{14.1}$$

$$\overline{\rho_1^{A}} = tr_B \overline{\rho_1^{AB}} = \sum_{\alpha\beta} \rho^{AB}_{\alpha\alpha\beta\beta} |\alpha\rangle\langle\alpha|, \tag{14.2}$$

$$\overline{\rho_1^{B}} = tr_A \overline{\rho_1^{AB}} = \sum_{\alpha\beta\beta'} \rho^{AB}_{\alpha\alpha\beta\beta'} |\beta\rangle\langle\beta'|, \tag{14.3}$$

$$\widetilde{\rho^{AB}} = \sum_{\beta} \Pi_\beta (\overline{\rho_1^{AB}}) \Pi_\beta = \sum_{\alpha\beta} \rho^{AB}_{\alpha\alpha\beta\beta} |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|, \tag{15.1}$$

$$\widetilde{\rho^{A}} = tr_B \widetilde{\rho^{AB}} = \sum_{\alpha\beta} \rho^{AB}_{\alpha\alpha\beta\beta} |\alpha\rangle\langle\alpha|, \tag{15.2}$$

$$\widetilde{\rho^{B}} = tr_A \widetilde{\rho^{AB}} = \sum_{\alpha\beta} \rho^{AB}_{\alpha\alpha\beta\beta} |\beta\rangle\langle\beta|. \tag{15.3}$$

Where $\rho^{AB}_{\alpha\alpha'\beta\beta'} = \langle\alpha\beta|\rho^{AB}|\alpha'\beta\rangle$. From Eq.(13.3) and Eq.(14.3), Eq.(14.2) and Eq.(15.2), we have

$$\rho^{B} = \overline{\rho_1^{B}}, \quad \overline{\rho_1^{A}} = \widetilde{\rho^{A}}. \tag{16}$$

Then

$$\begin{aligned}
& [S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^{A}}) - S(\widetilde{\rho^{B}})] - [S(\rho^{AB}) - S(\rho^{A}) - S(\rho^{B}))] \\
= \; & \{[S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^{A}}) - S(\widetilde{\rho^{B}})] - [S(\overline{\rho_1^{AB}}) - S(\overline{\rho_1^{A}}) - S(\overline{\rho_1^{B}})]\} \\
& + \{[S(\overline{\rho_1^{AB}}) - S(\overline{\rho_1^{A}}) - S(\overline{\rho_1^{B}})] - [S(\rho^{AB}) - S(\rho^{A}) - S(\rho^{B}))]\} \\
= \; & \{[S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^{B}})] - [S(\overline{\rho_1^{AB}}) - S(\overline{\rho_1^{B}})]\} + \{[S(\overline{\rho_1^{AB}}) - S(\overline{\rho_1^{A}})] - [S(\rho^{AB}) - S(\rho^{A})]\}.
\end{aligned} \tag{17}$$

From Eq.(2) and Eq.(3), it can be seen that the two expressions in the two curly braces of last line in Eq.(17) are both nonnegative, then we obtain Eq.(9).

To prove Eq.(10), suppose $D_{AB}(\rho^{AB}) = 0$ and the zero can be achieved by the two-sided projective measurement $\{\Pi_{\alpha\beta}\}_{\alpha\beta}$. Again from Eq.(17), it follows that the two expressions in the two curly braces of last line in Eq.(17) are both vanishing. Then $D_A(\rho^{AB}) = 0$ and $D_B(\overline{\rho^{AB}}) = 0$. Similarly, when we repeat the above program substituting $\overline{\rho_2^{AB}}$ by $\overline{\rho_2^{AB}} = \sum_\beta I_A \otimes \Pi_\beta (\rho^{AB}) I_A \otimes \Pi_\beta$,

we will obtain $D_B(\rho^{AB}) = 0$. Combining $D_A(\rho^{AB}) = 0$, $D_B(\rho^{AB}) = 0$, and Eq.(4), we steadily obtain Eq.(10). That is to say,

$$D_{AB}(\rho^{AB}) = 0 \Longleftrightarrow D_A(\rho^{AB}) = D_B(\rho^{AB}) = 0. \tag{18}$$

We then complete this proof.

The intuitive meaning of Eq.(8) is that $D_{AB}(\rho^{AB})$ is the minimal loss of mutual information over all two-sided projective measurements. From Eq.(3) and Eq,(10), or from Eq.(18), we see that $D_{AB}(\rho^{AB})$ captures more correlation than $D_A(\rho^{AB})$ in the sense

$$D_{AB}(\rho^{AB}) = 0 \Rightarrow D_A(\rho^{AB}) = 0. \tag{19}$$

Similar to Eq.(6), we also define a geometric measure of quantum discord over two-sided projective measurements as

$$D^G_{AB}(\rho^{AB}) = \inf_{\chi^{AB}} ||\rho^{AB} - \chi^{AB}||^2, \tag{20}$$

where $||\rho^{AB} - \chi^{AB}||^2 = tr[(\rho^{AB} - \chi^{AB})^2]$, inf takes all χ^{AB} that $D_{AB}(\chi^{AB}) = 0$. From the definitions of $D^G_A(\rho^{AB})$ and $D^G_{AB}(\rho^{AB})$, and Eq. (19), it can be easily found that

$$D^G_{AB}(\rho^{AB}) \geq \max\{D^G_A(\rho^{AB}), D^G_B(\rho^{AB})\}. \tag{21}$$

4. Simplification and a lower bound of Eq.(20)

Theorem 2 below will simplify Eq. (20).

Theorem 2. $D^G_{AB}(\rho^{AB})$ is defined in Eq. (20), then

$$\begin{aligned} D^G_{AB}(\rho^{AB}) &= \inf_{\{\Pi_{\alpha\beta}\}_{\alpha\beta}} ||\rho^{AB} - \sum_{\alpha\beta} \Pi_{\alpha\beta}\rho^{AB}\Pi_{\alpha\beta}||^2 & (22)\\ &= tr[(\rho^{AB})^2] - \sup_{\{\Pi_{\alpha\beta}\}_{\alpha\beta}} ||\sum_{\alpha\beta} \Pi_{\alpha\beta}\rho^{AB}\Pi_{\alpha\beta}||^2, & (23)\end{aligned}$$

where inf and sup take over all two-sided projective measurements $\{\Pi_{\alpha\beta}\}_{\alpha\beta}$.

Proof. For any χ^{AB} that $D_{AB}(\chi^{AB}) = 0$, suppose

$$\chi^{AB} = \sum_{\alpha\beta} p_{\alpha\beta} |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|. \tag{24}$$

Where $\{|\alpha\rangle\}_{\alpha=1}^{n_A}$, $\{|\beta\rangle\}_{\beta=1}^{n_B}$ are orthonormal bases for H^A and H^B, $p_{\alpha\beta} \geq 0$, $\sum_{\alpha\beta} p_{\alpha\beta} = 1$. We expand ρ^{AB} in the bases $\{|\alpha\rangle\}_{\alpha=1}^{n_A} = \{|\alpha'\rangle\}_{\alpha'=1}^{n_A}$ and $\{|\beta\rangle\}_{\beta=1}^{n_B} = \{|\beta'\rangle\}_{\beta'=1}^{n_B}$ as

$$\rho^{AB} = \sum_{\alpha\alpha'\beta\beta'} \rho^{AB}_{\alpha\alpha'\beta\beta'} |\alpha\rangle\langle\alpha'| \otimes |\beta\rangle\langle\beta'|. \tag{25}$$

where $\rho^{AB}_{\alpha\alpha'\beta\beta'} = \langle\alpha\beta|\rho^{AB}|\alpha'\beta'\rangle$, and $\rho^{AB}_{\alpha\alpha\beta\beta} = \langle\alpha\beta|\rho^{AB}|\alpha\beta\rangle \geq 0$, $\sum_{\alpha\beta} \rho^{AB}_{\alpha\alpha\beta\beta} = 1$. Consequently,

$$\begin{aligned} & ||\rho^{AB} - \chi^{AB}||^2 \\ = \;& tr[(\rho^{AB})^2] - 2\sum_{\alpha\beta} p_{\alpha\beta}\rho^{AB}_{\alpha\alpha\beta\beta} + \sum_{\alpha\beta} p^2_{\alpha\beta} \\ = \;& tr[(\rho^{AB})^2] - \sum_{\alpha\beta}(\rho^{AB}_{\alpha\alpha\beta\beta})^2 + \sum_{\alpha\beta}(p_{\alpha\beta} - \rho^{AB}_{\alpha\alpha\beta\beta})^2. \qquad (26)\end{aligned}$$

By choosing $p_{\alpha\beta} = \rho^{AB}_{\alpha\alpha\beta\beta}$, i.e., $\chi^{AB} = \Sigma_{\alpha\beta}\Pi_{\alpha\beta}\rho^{AB}\Pi_{\alpha\beta}$, we then attain Theorem 3.

We would rather like to give another expression of Theorem 2, a lower bound of $D^G_{AB}(\rho^{AB})$ will follow from it, that is Theorem 3 below.

Theorem 3. $D^G_{AB}(\rho^{AB})$ is defined in Eq. (20), then

$$D^G_{AB}(\rho^{AB}) = tr(CC^t) - \sup_{AB} tr(ACB^tBC^tA^t), \tag{27}$$

$$D^G_{AB}(\rho^{AB}) \geq tr(CC^t) - \sum_{k=1}^{\min\{n_A,n_B\}} \lambda_k. \tag{28}$$

Where λ_k are the eigenvalues of CC^t listed in decreasing order (counting multiplicity), t denotes transpose. Real matrices A, B, C are specified as follows: given orthonormal bases $\{X_i\}_{i=1}^{n_A^2}$ for $L(H^A)$ and $\{Y_j\}_{j=1}^{n_B^2}$ for $L(H^B)$. Let $\rho^{AB} = \sum_{ij} C_{ij} X_i \otimes Y_j$, then matrix $C = (C_{ij})$. For any orthonormal bases $\{|\alpha\rangle\}_{\alpha=1}^{n_A}$ for H^A and $\{|\beta\rangle\}_{\beta=1}^{n_B}$ for H^B, let $|\alpha\rangle\langle\alpha| = \sum_{i=1}^{n_A^2} A_{\alpha i}X_i$, $|\beta\rangle\langle\beta| = \sum_{j=1}^{n_B^2} B_{\beta j}Y_j$, then matrices $A = (A_{\alpha i})$, $B = (B_{\beta j})$.

To prove Eq. (27), note that $A_{\alpha i} = tr(X_i|\alpha\rangle\langle\alpha|) = \langle\alpha|X_i|\alpha\rangle$, $B_{\beta j} = tr(Y_j|\beta\rangle\langle\beta|) = \langle\beta|Y_j|\beta\rangle$, thus

$$\begin{aligned}
&D^G_{AB}(\rho^{AB}) = tr[(\rho^{AB})^2] - \sup_{\{\Pi_{\alpha\beta}\}} \|\sum_{\alpha\beta} \Pi_{\alpha\beta}\rho^{AB}\Pi_{\alpha\beta}\|^2 \\
&= \sum_{ij} C_{ij}^2 - \sup_{\{\Pi_{\alpha\beta}\}} \|\sum_{ij\alpha\beta} C_{ij}\langle\alpha|X_i|\alpha\rangle\langle\beta|Y_j|\beta\rangle|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|\|^2 \\
&= \sum_{ij} C_{ij}^2 - \sup_{AB} \sum_{\alpha\beta}(\sum_{ij} A_{\alpha i}C_{ij}B_{\beta j})^2 \\
&= tr(CC^t) - \sup_{AB} tr(ACB^tBC^tA^t).
\end{aligned}$$

A brief proof of inequality (28) is: since [26]

$$D^G_A(\rho^{AB}) \geq tr(CC^t) - \sum_{k=1}^{n_A} \lambda_k, \tag{29}$$

$$D^G_B(\rho^{AB}) \geq tr(CC^t) - \sum_{k=1}^{n_B} \lambda_k, \tag{30}$$

together with Eq.(21), so inequality (28) is surely true.

5. Examples

Let us consider some examples which allow explicit results for $D^G_{AB}(\rho^{AB})$.

Example 1. For the $m \times m$ Werner state

$$\rho^{AB} = \frac{m-x}{m^3-m} I_{AB} + \frac{mx-1}{m^3-m} F, \quad x \in [-1, 1], \tag{31}$$

with $F = \sum_{kl} |k\rangle\langle l| \otimes |l\rangle\langle k|$, $\{|k\rangle\} = \{|l\rangle\}$ is an orthonormal basis for H^A ($H^A = H^B$). Note that $F^2 = I_{AB}$, $trF = m$. For any two-sided projective measurement $\{\Pi_{\alpha\beta}\}_{\alpha\beta}$,

$$\sum_{\alpha\beta} \Pi_{\alpha\beta}\rho^{AB}\Pi_{\alpha\beta}$$

$$
\begin{aligned}
&= \frac{m-x}{m^3-m} I_{AB} + \frac{mx-1}{m^3-m} \sum_{\alpha\beta kl} \langle\alpha|k\rangle\langle l|\alpha\rangle\langle\beta|l\rangle\langle k|\beta\rangle\Pi_{\alpha\beta} \\
&= \frac{m-x}{m^3-m} I_{AB} + \frac{mx-1}{m^3-m} \sum_{\alpha\beta} |\langle\alpha|\beta\rangle|^2\Pi_{\alpha\beta}.
\end{aligned}
$$

By Eq. (22), and applying the Lagrangian multipliers method, we get

$$
D^G_{AB}(\rho^{AB}) = \frac{(mx-1)^2}{m(m-1)(m+1)^2}. \tag{32}
$$

That is $D^G_{AB}(\rho^{AB}) = D^G_A(\rho^{AB})$ [26].

A Werner state is separable if and only if $x \in [0,1]$ [2], but $D^G_A(\rho^{AB}) = D^G_{AB}(\rho^{AB}) = 0$ if and only if $x = 1/m$, i.e., it is the completely mixed state.

Example 2. For the $m \times m$ isotropic state

$$
\rho^{AB} = \frac{1-x}{m^2-1} I_{AB} + \frac{m^2x-1}{m^2-1} M, \quad x \in [0,1], \tag{33}
$$

with $M = \frac{1}{m}\sum_{kl}|k\rangle\langle l| \otimes |k\rangle\langle l|$, $\{|k\rangle\} = \{|l\rangle\}$ is an orthonormal basis for H^A ($H^A = H^B$). Note that $M^2 = M$, and $trM = 1$. For any two-sided projective measurement $\{\Pi_{\alpha\beta}\}_{\alpha\beta}$

$$
\begin{aligned}
&\sum_{\alpha\beta} \Pi_{\alpha\beta}\rho^{AB}\Pi_{\alpha\beta} \\
&= \frac{1-x}{m^2-1} I_{AB} + \frac{m^2x-1}{m^2-1}\frac{1}{m} \sum_{\alpha\beta kl} \langle\alpha|k\rangle\langle l|\alpha\rangle\langle\beta|k\rangle\langle l|\beta\rangle\Pi_{\alpha\beta} \\
&= \frac{1-x}{m^2-1} I_{AB} + \frac{m^2x-1}{m^2-1}\frac{1}{m} \sum_{\alpha\beta} |\langle\alpha|\beta'\rangle|^2\Pi_{\alpha\beta}.
\end{aligned}
$$

here $|\beta'\rangle = |\beta\rangle^*$ is the complex conjugate of $|\beta\rangle$ in the basis $\{|k\rangle\} = \{|l\rangle\}$, namely, $\langle\beta|k\rangle = \langle k|\beta'\rangle$, $\langle l|\beta\rangle = \langle\beta'|l\rangle$. Using the similar techniques in example 1 we get

$$
D^G_{AB}(\rho^{AB}) = \frac{(m^2x-1)^2}{m(m-1)(m+1)^2}. \tag{34}
$$

That is $D^G_{AB}(\rho^{AB}) = D^G_A(\rho^{AB})$ [26].

Recall that an isotropic state is separable if and only if $x \in [0, 1/m]$ [2], but $D^G_A(\rho^{AB}) = D^G_{AB}(\rho^{AB}) = 0$ if and only if $x = 1/m^2$, i.e., it is the completely mixed state.

Example 3. For any two-qubit state

$$
\begin{aligned}
\rho &= \frac{1}{4}(I_{AB} + \sum_{i=1}^{3} x_i\sigma_i \otimes I_B + \sum_{j=1}^{3} y_j I_A \otimes \sigma_j + \sum_{i,j=1}^{3} T_{ij}\sigma_i \otimes \sigma_j) \\
&= \frac{1}{2}(X_0 \otimes Y_0 + \sum_{i=1}^{3} x_i X_i \otimes Y_0 + \sum_{j=1}^{3} y_j X_0 \otimes Y_j + \sum_{i,j=1}^{3} T_{ij} X_i \otimes Y_j).
\end{aligned} \tag{35}
$$

Where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ are two real vectors, $\{\sigma_i\}$ are the Pauli matrices, $\{X_0, X_1, X_2, X_3\} = \{I_A, \sigma_1, \sigma_2, \sigma_3\}/\sqrt{2}$, $\{Y_0, Y_1, Y_2, Y_3\} = \{I_B, \sigma_1, \sigma_2, \sigma_3\}/\sqrt{2}$. Note that $tr\sigma_i = 0$

and $tr(\sigma_i\sigma_j) = 2\delta_{ij}$, hence $\{X_0, X_1, X_2, X_3\}$ is an orthonormal basis for $L(H^A)$, and $\{Y_0, Y_1, Y_2, Y_3\}$ is an orthonormal basis for $L(H^B)$. For any orthonormal basis $\{|\alpha\rangle\}_{\alpha=1}^2$ of H^A, $|\alpha\rangle\langle\alpha| \in L(H^A)$, we can write $|\alpha\rangle\langle\alpha|$ as

$$|1_A\rangle\langle 1_A| = (X_0 + a_1X_1 + a_2X_2 + a_3X_3)/\sqrt{2}, \tag{36.1}$$

$$|2_A\rangle\langle 2_A| = (X_0 - a_1X_1 - a_2X_2 - a_3X_3)/\sqrt{2}. \tag{36.2}$$

Here, $\mathbf{a} = (a_1, a_2, a_3)$ is a real vector with $||\mathbf{a}||^2 = \sum_{i=1}^3 a_i^2 = 1$. Similarly, for any orthonormal basis $\{|\beta\rangle\}_{\alpha=1}^2$ of H^B, $|\beta\rangle\langle\beta| \in L(H^B)$, we write $|\beta\rangle\langle\beta|$ as

$$|1_B\rangle\langle 1_B| = (Y_0 + b_1Y_1 + b_2Y_2 + b_3Y_3)/\sqrt{2}, \tag{37.1}$$

$$|2_B\rangle\langle 2_B| = (Y_0 - b_1Y_1 - b_2Y_2 - b_3Y_3)/\sqrt{2}. \tag{37.2}$$

Here, $\mathbf{b} = (b_1, b_2, b_3)$ is a real vector with $||\mathbf{b}||^2 = \sum_{i=1}^3 b_i^2 = 1$. Thus, from Eq. (27), direct calculation shows that

$$D_{AB}^G(\rho^{AB}) = \frac{1}{4}[||\mathbf{x}||^2 + ||\mathbf{y}||^2 + tr(TT^t)] - \frac{1}{4}\sup_{\mathbf{ab}}[(\mathbf{a}\cdot\mathbf{x})^2 + (\mathbf{b}\cdot\mathbf{y})^2 + (\mathbf{a}T\mathbf{b}^t)^2]. \tag{38}$$

Where $\mathbf{a}\cdot\mathbf{x} = \sum_{i=1}^3 a_ix_i$, $\mathbf{a}\cdot\mathbf{y} = \sum_{i=1}^3 a_iy_i$, $T = (T_{ij})$. It is desirable but seems not easy to optimize Eq. (38), here we only discuss some special cases of it:

(i) if $T = 0$, then $D_{AB}^G(\rho^{AB}) = 0$;

(ii) if $\mathbf{x} = \mathbf{y} = 0$, that is $\rho^A = I_A$ and $\rho^B = I_B$, by the singular value decomposition of T, we get $D_{AB}^G(\rho^{AB}) = \frac{1}{4}[tr(TT^t) - \lambda_{\max}]$, with $\lambda_{\max}$ being the largest eigenvalue of TT^t;

(iii) if $T_{ij} = x_iy_j$, that is $\rho^{AB} = \rho^A \otimes \rho^B$, then $D_{AB}^G(\rho^{AB}) = 0$.

6. Summary

We introduced a geometric measure for the quantum discord defined over two-sided projective measurements, simplified the expression and provided a lower bound for this geometric measure. Some special quantum states were discussed as demonstrations of this geometric measure. We expect that this geometric measure may provide an new perspective and bring some conveniences for understanding and characterization of quantum discord over two-sided projective measurements.

It has shown that $D_{AB}(\rho^{AB})$ captures more correlation than $D_A(\rho^{AB})$. At the end of this paper, it is interesting to point out the ordering of some different quantum correlation below

$$\rho^{AB} = \rho^A \otimes \rho^B \Longrightarrow D_{AB}(\rho^{AB}) = 0 \Longrightarrow D_A(\rho^{AB}) = 0 \Longrightarrow \rho^{AB} \textit{ is separable.}$$

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